But now, if this however great KB is supposed divided (as in Proposition XXV) into portions KK, equal to the length R, and from these points K perpendiculars are erected, which meet AX in points H, D, M ; the angles at these points, toward the parts of the point L, will neither be right nor obtuse ; lest in some quadrilateral, as suppose $KMLK$, the four angles together should be equal to or greater than four rights, contrary to the hypothesis of acute angle, according to which we are proceeding. Therefore all such angles will be acute toward the parts of the point L ; and therefore in like manner all at these points obtuse toward the parts of the point A. Wherefore (from Corollary I to Proposition III) of the aforesaid perpendiculars the least will indeed be KL more remote from the base AB, the greatest KM nearer this base.

And of the remaining the nearer will be ever greater than the more remote.

Therefore (from the preceding Proposition XXV, and its Corollary) the four angles together of the quadrilateral $KHLK$ more remote from base AB will be greater than the four angles together of all the remaining quadrilaterals nearer to this base. Wherefore (as in Proposition XXV) the hypothesis of acute angle would be destroyed.

Therefore it holds, that of the aforesaid common perpendiculars in two distinct points there will be no determinate limit, such that under a smaller and smaller acute angle made at the point A, it would not always be possible to attain (in hypothesis of acute angle) to such a common perpendicular in two distinct points as may be less than any assigned length R.

Quod erat demonstrandum.

[To be Continued.]

SOPHUS LIE'S TRANSFORMATION GROUPS.

A SERIES OF ELEMENTARY, EXPOSITORY ARTICLES.

By EDGAR ODELL LOVETT, Princeton, New Jersey.

IV.

Proof of Lie's theorem that a one parameter group contains but one infinitesimal transformation and its converse theorem. Examples.

13. In the preceding paragraphs it has been shown by methods of proof due to Lie that every one parameter group with inverse transformations contains an infinitesimal transformation and conversely, every infinitesimal transformation generates a one parameter group. It is the purpose of this paragraph to present the proof of the theorem that the indefinite article "a" in these theorems can be replaced by the definite modifier "one and but one." The theorem

is necessary to a rigorous grounding of the fundamental details of the theory of the group of one parameter ; the proof is less simple than the proofs of the previous theorems because it makes use of an elementary theorem of the theory of functions.

Consider again the G_1

$$x_1=\varphi(x, y, a), \quad y_1=\psi(x, y, a). \tag{1}$$

We have found that every G_1 contains an infinitesimal transformation ; hence we may assume the existence of an infinitesimal transformation, say

$$x'=x+\xi(x, y)\delta t+\ldots\ldots, \quad y'=y+\eta(x, y)\delta t+\ldots\ldots, \tag{2}$$

belonging to the G_1, (1).

Let T_a be the transformation of the group (1) corresponding to the value a of the parameter and carrying the point (x, y) into the position (x_1, y_1). Let I be the infinitesimal transformation (2) of the group (1) and let it change the point (x_1, y_1) into the point (x_2, y_2) given by the equations

$$I \qquad x_2=x_1+\xi(x_1, y_1)\delta t+\ \ldots\ldots, \quad y_2=y_1+\eta(x_1, y_1)\delta t+\ldots\ldots. \tag{3}$$

The transformation equivalent to the product T_aI transforms the point (x, y) into the point (x_2, y_2) and is found by eliminating (x_1, y_1) from the equations (3) by means of the equations (1). This elimination partially effected gives

$$S\equiv T_aI\equiv T_{a+\delta a}\begin{cases} x_2=\varphi(x, y, a)+\xi(x_1, y_1)\delta t+\ldots\ldots, \\ y_2=\psi(x, y, a)+\eta(x_1, y_1)\delta t+\ldots\ldots, \end{cases} \tag{4}$$

where the x_1 and y_1 allowed to remain are to be expressed in terms of x, y, and a by means of the equations (1).

The equations (4) represent the transformation S which is equivalent to the successive application of T_a and I. Since T_a and I belong to the group (1), their product S belongs, by definition, to the group (1). Since I is an infinitesimal, the product, S, of T_a and I differs by an infinitesimal from T_a and accordingly the parameter of S has a value, $a+\delta a$, differing by an infinitesimal from the parameter, a, of T_a, where δa is an infinitesimal quantity of the same nature as δt in the equations (2). Further, we have proved that if a is the parameter of a transformation of a given group and a_1 the parameter of a second transformation of the same group, the parameter α of their product is a function of a and a_1 alone. The parameter of T_a is a, that of I is δt, and that of S, the product of T_a and I, is $a+\delta a$; hence $a+\delta a$ is a function of a and δt alone, *i. e.* δa depends on a and δt alone.

The transformation $T_{a+\delta a}$ which is equivalent to S and is a member of the group (1) has the form

$$x_2=\varphi(x, y, a+\delta a), \quad y_2=\psi(x, y, a+\delta a), \tag{5}$$

or developed in powers of δa,

$$x_2=\varphi(x, y, a)+\frac{\partial\varphi(x, y, a)}{\partial a}\delta a+\ldots\ldots,$$
$$y_2=\psi(x, y, a)+\frac{\partial\psi(x, y, a)}{\partial a}\delta a+\ldots\ldots. \tag{6}$$

Comparing these expressions (6) for x_2, y_2 with the forms given by the equations (4) we have

$$\left\{\begin{aligned}&\xi(x_1, y_1)\delta t+\ldots\ldots=\frac{\partial\varphi(x, y, a)}{\partial a}\delta a+\ldots\ldots,\\ &\eta(x_1, y_1)\delta t+\ldots\ldots=\frac{\partial\psi(x, y, a)}{\partial a}\delta a+\ldots\ldots.\end{aligned}\right. \tag{7}$$

In these equations (7), x_1 and y_1 are definite functions of x, y, and a, and conversely, x and y are definite functions of x_1, y_1, and a, given by the equations (1) ; δa and δt are infinitesimals and, as is shown above, δa is a function of a and δt ; the equations are true for all values of x, y, and a.

If in the equations (1) x and y are given any definite numerical values, x_1 and y_1 depend only on a. Hence if x and y are given any definite numerical values in the equations (7), these equations will express relations in δt, δa and a alone. These relations must of course agree with the one that expresses δa in terms of a and δt. Now we can choose the numbers x_1 and y_1 so that the first coefficients of one of the two relations, namely the quaatities

$$\xi(x_1, y_1), \quad \frac{\partial\varphi(x, y, a)}{\partial a}, \tag{8}$$

or the quantities

$$\eta(x_1, y_1), \quad \frac{\partial\psi(x, y, a)}{\partial a}, \tag{9}$$

do not vanish.*

Then the first or second equation (7) gives the relation between δa, δt and a in the form

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*It is obvious that both of the quantities $\xi(x_1, y_1)$ $\eta(x_1, y_1)$ cannot vanish identically, else we should have no transformation. We may assume then that $\xi(x_1, y_1)$ does not vanish. It is clear then that $\frac{\partial\varphi(x, y, a)}{\partial a}$ cannot, in general, vanish ; for if it should vanish identically for all values of x and y, $\varphi(x, y, a)$ would be free from a and hence x would not be transformed at all, *i. e.* $\xi(x_1, y_1)$ would be identically zero ; but the latter is contrary to hypothesis.

$$u_1\delta t+u_2\delta t^2+\ldots\ldots=v_1\delta a+v_2\delta a^2+\ldots\ldots, \tag{10}$$

in which $u_1, u_2, \ldots\ldots, v_1, v_2, \ldots\ldots$ depend upon a. u_1, v_1 are both different from zero since they replace the quantities (8) or the quantities (9).

Now it is a theorem of the theory of functions that when two quantities δa and δt are related as in (10), δa may be developed in a power series of δt, whose first coefficient is not zero, *i. e.*

$$\delta a=w_1\delta t+w_2\delta t^2+\ldots\ldots,$$

where w_1 is not identically zero. The w_i are certain functions of a. Substituting this value of δa in the equations (7), conceiving x_1 and y_1 as variables again, we have

$$\left\{\begin{aligned}&\xi(x_1, y_1)\delta t+\ldots\ldots=\frac{\partial\varphi(x, y, a)}{\partial a}(w_1\delta t+w_2\delta t^2+\ldots\ldots)+\ldots\ldots,\\&\eta(x_1, y_1)\delta t+\ldots\ldots=\frac{\partial\psi(x, y, a)}{\partial a}(w_1\delta t+w_2\delta t^2+\ldots\ldots)+\ldots\ldots.\end{aligned}\right. \tag{11}$$

Dividing through by δt and passing to the limit $\delta t=0$, these become

$$\left\{\begin{aligned}&\xi(x_1, y_1)=\frac{\partial\varphi(x, y, a)}{\partial a}w_1(a),\\&\eta(x_1, y_1)=\frac{\partial\psi(x, y, a)}{\partial a}w_1(a),\end{aligned}\right. \tag{12}$$

The equations (1) solved for x and y give

$$x=\lambda(x_1, y_1, a),\quad y=\mu(x_1, y_1, a). \tag{13}$$

If these values of x and y be put in the equations (12) we have

$$\left\{\begin{aligned}&\xi(x_1, y_1)=X(x_1, y_1, a)w_1(a),\\&\eta(x, y_1)=Y(x_1, y_1, a)w_1(a).\end{aligned}\right. \tag{14}$$

The equations (14) must be true for all valoes of x_1, y_1 and a. Their left members do not contain a, hence their right members do not really contain a, but only in appearance, *i. e.* the functions X and Y have the form

$$X(x_1, y_1, a)\equiv\frac{A(x_1, y_1)}{w(a)},\qquad Y(x_1, y_1, a)\equiv\frac{B(x_1, y_1)}{w(a)}. \tag{15}$$

If we give now to the quantity a in the equations (14) a definite value* $\bar{a}$, the functions X and Y are changed into functions of x_1 and y_1 alone,

*For example, a solution of the equation $w(a)-1=0$ would be such a value of a that would reduce the functions X and Y to functions of x_1 and y_1 alone.

$$X(x_1, y_1, \bar{a}) \equiv \bar{X}(x_1, y_1), \quad Y(x_1, y_1, \bar{a}) \equiv \bar{Y}(x_1, y_1). \tag{16}$$

$w_1(a)$ becomes $w_1(\bar{a})$, which is a constant, since $\bar{a}$ is a definite number. Then, excepting this constant factor, the defining functions $\xi(x_1, y_1)$ and $\eta(x_1, y_1)$ of the infinitesimal transformation are completely determined, that is, if we call this constant k, we have

$$\xi(x_1, y_1) = k\bar{X}(x_1, y_1), \quad \eta(x_1, y_1) = k\bar{Y}(x_1, y_1). \tag{17}$$

This result obtains for *every* infinitesimal transformation (2) of the G_1 (1).

Any two infinitesimal transformations of the group (1), say

$$x' = x + \xi\delta t + \ldots\ldots, \quad y' = y + \eta\delta t + \ldots\ldots$$

and

$$x' = x + \bar{\xi}\delta t + \ldots\ldots, \quad y' = y + \bar{\eta}\delta t + \ldots\ldots \tag{18}$$

can differ in their terms of the first order only by a constant factor, since

$$\bar{\xi} \equiv k\xi, \quad \bar{\eta} \equiv k\eta.$$

LIE calls two such infinitesimal transformations, whose terms of the first order differ only by a constant factor, *dependent* infinitesimal transformations since they are essentially the same, for, inasmuch as δt was taken as an arbitrary infinitely small quantity, $k\delta t$ has the same meaning as δt.

In this manner LIE arrives at the theorem: *A one parameter group of the plane with inverse transformations contains one infinitesimal transformation and no more; or, more accurately expressed, all infinitesimal transformations of a one parameter group agree in their terms of the first order excepting a constant factor.**

14. The factor $w_1(a)$ may be gotten rid of in the equations (12) by introducing a new parameter in the group (1) in place of a. If a_0 is the value of the parameter a which gives the identical transformation of the group (1), the new parameter, t, is given by

$$t = \int_{a_0}^{a} \frac{da}{w_1(a)}, \tag{19}$$

and the equations of the group become

$$x_1 = \Phi(x, y, t), \quad y_1 = \Psi(x, y, t). \tag{20}$$

Since $a = a_0$ makes $t = 0$ in (19), it is clear that in the form (20) the identical transformation, $x_1 = x$, $y_1 = y$, is given by the value of the parameter $t = 0$. Hence the equations (12) may be written

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*See LIE—*Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen*, bearbeitet und herausgegeben von Dr. Georg Scheffers, Leipzig, 1891, pp. 38 et seq.

$$\left\{\begin{aligned} \xi(x_1, y_1) &= \frac{\partial\varphi(x, y, a)}{\partial a}\,\frac{da}{dt} \equiv \frac{\partial\Phi(x, y, t)}{\partial t}, \\ \eta(x_1, y_1) &= \frac{\partial\psi(x, y, a)}{\partial a}\,\frac{da}{dt} \equiv \frac{\partial\Psi(x, y, t)}{\partial t}, \end{aligned}\right. \tag{21}$$

or

$$\xi(x_1, y_1) = \frac{dx_1}{dt}, \quad \eta(x_1, y_1) = \frac{dy_1}{dt}, \tag{22}$$

since x_1, y_1 are the functions (1) of a or the functions (20) of t.

Hence the equations (20), which are equivalent to the original equations (1) of the group, are the integral equations of the simultaneous system

$$\frac{dx_1}{\xi(x_1, y_1)} = \frac{dy_1}{\eta(x_1, y_1)} = dt, \tag{23}$$

with the initial conditions $x_1 = x$, $y_1 = y$, $t = 0$.

Since this simultaneous system and its integral equations are completely determined when the first members

$$x' = x + \xi(x, y)\delta t, \quad y' = y + \eta(x, y)\delta t,$$

of the infinitesimal transformation of the group are given, Lie has the theorem—*a one parameter group of the plane is completely defined by its infinitesimal transformation; or otherwise expressed, every infinitesimal transformation belongs to one one parameter group of the plane and no more.*

This theorem and the preceding one are incorporated in the following second fundamental theorem of Lie's theory of the group of one parameter.

Theorem: *Every one parameter group of the plane whose transformations are inverse in pairs contains one infinitesimal transformation and no more. Every infinitesimal transformation of the plane belongs to one one parameter group and to one only. The latter's transformations are inverse in pairs.*

The reader may find it to his advantage to work through the following examples carefully. They are in illustration of the theorems of the last two notes of this series and are drawn from Lie's lectures on differential equations.

1°. If the given infinitesimal transformation has the form

$$x_1 = x + x\delta t, \quad y_1 = y + y\delta t,$$

then

$$\xi(x, y) \equiv x, \quad \eta(x, y) \equiv y,$$

and the simultaneous system is

$$\frac{dx_1}{x_1} = \frac{dy_1}{y_1} = dt.$$

The integration of this simultaneous system gives

$$\log x_1 - \log x = \log y_1 - \log y = t,$$

or
$$x_1 = e^t x, \quad y_1 = e^t y.$$

These equations represent a one parameter group as may be readily verified directly. Any transformation of the group changes all abscissas and ordinates in the same ratio, *i. e.* the entire plane is proportionately increased or diminished from the origin of coördinates. $t=0$ gives the infinitesimal transformation of the group. By the exponential theorem we have

$$e^{\delta t} = 1 + \frac{\delta t}{1\,!} + \frac{\delta t^2}{2\,!} + \ldots\ldots;$$

hence the infinitesimal transformation of the group is

$$x_1 = (1 + \frac{\delta t}{1\,!} + \frac{\delta t^2}{2\,!} + \ldots\ldots)x, \quad y_1 = (1 + \frac{\delta t}{1\,!} + \frac{\delta t^2}{2\,!} + \ldots\ldots)y,$$

which agrees, in its terms of the first order, with the original infinitesimal transformation.

2°. The equations

$$x_1 = \sqrt{x^2 + xyt}, \quad y_1 = \frac{xy}{\sqrt{x^2 + xyt}}$$

represent a one parameter group, as appears in the following manner by making use again of the fundamental theorem of note III. We have clearly

$$x_1 y_1 = xy \quad \text{and} \quad \frac{x_1}{y_1} = \frac{x^2 + xyt}{xy} \equiv \frac{x}{y} + t.$$

These equations have the form of those in the theorem named if we put

$$\Omega(x, y) \equiv xy, \quad W(x, y) \equiv x/y.$$

$t=0$ corresponds to the identical transformation of the group, hence the infinitesimal transformation has the parameter $t=\delta t$. Since

$$\sqrt{x^2 + xyt} = x\sqrt{1 + \frac{y}{x}\delta t} = x(1 + \tfrac{1}{2}\frac{y}{x}\delta t + \ldots\ldots),$$

the infinitesimal transformation of the group is represented by tne equations

$$x_1 = x + \tfrac{1}{2} y \delta t + \ldots\ldots, \quad y_1 = y - \tfrac{1}{2}\frac{y^2}{x}\delta t + \ldots\ldots.$$

Hence,
$$\xi(x, y) \equiv \tfrac{1}{2} y, \quad \eta(x, y) \equiv -\tfrac{1}{2}\frac{y^2}{x},$$

and the finite equations of the group will appear again by integrating the simultaneous system.

$$\frac{2dx_1}{y_1} = -\frac{2x_1 dy_1}{y_1^2} = dt.$$

3°. x_1 and y_1, the roots u of the quadratic equation

$$(u-x)(u-y)+t=0,$$

expressed in terms of x, y and t, define a G_1.

The equation may be written

$$u^2-(x+y)u+xy+t=0.$$

Then by an elcmentary theorem of the theory of algebraic equations,

$$x_1+y_1=x+y,$$

$$x_1 y_1=xy+t.$$

These are two equations of the form

$$\Omega(x_1, y_1)=\Omega(x, y), \quad W(x_1, y_1)-t=W(x, y),$$

and hence represent a group.

By actually solving the equations it may readily be shown that thatt he infinitesimal transformation of the group is given by the equations

$$x_1=x+\frac{\delta t}{y-x}+\ldots\ldots, \quad y_1=y+\frac{\delta t}{x-y}+\ldots\ldots.$$

4°. The equations

$$x_1=x+t, \quad y_1=\frac{xy-t}{x+t}$$

represent a G_1. In this case

$$\xi(x, y)\equiv, \quad \eta(x, y)\equiv-\frac{1+y}{x}.$$

The finite equations are integral equations of the simultaneous system

$$\frac{dx_1}{1} = -\frac{x_1 dy_1}{1+y_1} = dt.$$

The reader will have no trouble in verifying these results.

Princeton University, 6 January, 1898.

[To be Continued.]